GeeksforGeeks

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DATA SCIENCE & ARTIFIACL INTELLIGENCE (DA)

LINEAR ALGEBRA

SHORT NOTES

Linear Algebra

1. Matrix

1.1 Matrices

A matrix is a rectangular arrangement of numbers, symbols, or expressions in rows and columns. It is generally denoted by capital letters (e.g., A, B, C) and represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Here:

- a_{ij} = element in the i^{th} row and j^{th} column.
- m = number of rows.
- n = number of columns.

1.2 Order of a Matrix

The order of a matrix refers to its dimensions:

$$\text{Order } = m \times n$$

where:

- m = number of rows
- n = number of columns

1.3 Operations on Matrices

1.3.1 Equality of Matrices

Two matrices A and B are equal if:

- They have the same order.
- Corresponding elements are equal: $a_{ij} = b_{ij}$.

1.3.2 Addition

If A and B are of same order:

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

1.3.3 Subtraction

If A and B are of same order:

$$(A - B)_{ij} = a_{ij} - b_{ij}$$

1.3.4 Scalar Multiplication

If k is a scalar:

$$(kA)_{ij} = k \cdot a_{ij}$$

1.3.5 Matrix Multiplication

If A is $m \times n$ and B is $n \times p$.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

1.4 Transpose of a Matrix

The transpose of a matrix A, denoted A^T, is obtained by interchanging rows and columns:

$$(A^T)_{ij} = a_{ji}$$

Let A and B be matrices of appropriate order, and k be a scalar.

1. Double Transpose
$$(A^T)^T = A$$
2. Transpose of a Sum
$$(A + B)^T = A^T + B^T$$
3. Transpose of a Difference
$$(A - B)^T = A^T - B^T$$
4. Transpose of a Scalar Multiple
$$(kA)^T = kA^T$$
5. Transpose of a Product
$$(AB)^T = B^T A^T$$
6. Transpose of an Inverse (if A is invertible)
$$(A^{-1})^T = (A^T)^{-1}$$

1.5 Trace of a Matrix

The trace of a square matrix A is the sum of its diagonal elements:

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Let A and B be square matrices of the same order, and k be a scalar.

1. Trace of a Scalar Multiple
$$\text{tr}(kA) = k \cdot \text{tr}(A)$$
2. Trace of a Sum
$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$
3. Trace of a Difference
$$\text{tr}(A - B) = \text{tr}(A) - \text{tr}(B)$$
4. Trace of a Product (Commutative Property for Trace)
$$\text{tr}(AB) = \text{tr}(BA)$$

(Note: This holds even if $AB \neq BA$.)

5. Cyclic Property for Multiple Matrices
$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$$
6. Trace of a Transpose
$$\text{tr}(A^T) = \text{tr}(A)$$

1.6 Types of Matrices

1.6.1 Row Matrix - Only one row.

1.6.2 Column Matrix - Only one column.

1.6.3 Square Matrix - Number of rows = number of columns.

1.6.4 Zero/Null Matrix - All elements are zero.

1.6.5 Identity Matrix - Square matrix with 1's on main diagonal and 0's elsewhere.

1.6.6 Diagonal Matrix - Non-zero elements only on the main diagonal.

1.6.7 Scalar Matrix - Diagonal matrix with equal diagonal elements.

1.6.8 Upper Triangular Matrix - All elements below main diagonal are zero.

1.6.9 Lower Triangular Matrix - All elements above main diagonal are zero.

1.6.10 Symmetric Matrix- $A^T = A$.

1.6.11 Skew-Symmetric Matrix $-A^T = -A$.

1.6.12 Orthogonal Matrix $-A^T A = I$.

1.6.13 Hermitian Matrix - $A^H = A$ (complex conjugate transpose).

1.6.14 Skew-Hermitian Matrix- $A^H = -A$.

1.6.15 Involutory Matrix $-A^2 = I$.

1.6.16 Idempotent Matrix - A matrix A is idempotent if $A^2 = A$.

1.6.17 Nilpotent Matrix - A matrix A is nilpotent if $A^k = 0$ for some positive integer k.

2. Determinant and its properties

2.1 Determinant

A determinant is a scalar value calculated from a square matrix.
It is denoted as $\det(A)$ or $|A|$.
For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

For a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Geometrical Meaning:

- In 2D, determinant represents the area of the parallelogram formed by column vectors.
- In 3D, determinant represents the volume of the parallelepiped formed by column vectors.

2.2 Properties of Determinants

1. For $n \times n$ matrix $A, \det(A^T) = \det(A)$
2. If two rows or columns of A are interchanged, $\det(A)$ changes sign
3. If two rows or columns of A are identical, $\det(A) = 0$
4. If a row or column of A is multiplied by k, determinant is multiplied by k
5. A common factor from any row or column of A can be taken outside $\det(A)$
6. If any row or column of A is all zeros, $\det(A) = 0$
7. Adding a multiple of one row/column to another in A does not change $\det(A)$
8. For $n \times n$ matrices A and $B, \det(AB) = \det(A) \cdot \det(B)$
9. For invertible $n \times n$ matrix $A, \det(A^{-1}) = \frac{1}{\det(A)}$
10. For triangular $n \times n$ matrix $A, \det(A) =$ product of diagonal elements
    2.3 Adjoint and Inverse of a Matrix
    2.3.1 Minor of an Element

For a square matrix $A = [a_{ij}]$, the minor of an element a_{ij} is the determinant of the submatrix obtained by deleting the i^{th} row and j^{th} column from A.

M_{ij}
$=$ determinant of submatrix after removing row i and column j.

2.3.2 Cofactor of an Element

The cofactor of a_{ij} is:

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Where:

- $(-1)^{i+j}$ introduces the alternating sign pattern.

2.3.3 Adjoint (Adjugate) of a Matrix

The adjoint (or adjugate) of A is the transpose of the cofactor matrix.

If:

$$\text{Cofactor Matrix of } A = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

Then:

$$\text{Adj}(A) = [C_{ij}]^T$$

2.3.4 Properties of Adjoint

1. $A \cdot \text{Adj}(A) = \text{Adj}(A) \cdot A = |A| \cdot I_n$
2. $\text{Adj}(A^T) = (\text{Adj}(A))^T$
3. If A is invertible, $\text{Adj}(A) \neq 0$.
4. $\text{Adj}(kA) = k^{n-1} \cdot \text{Adj}(A)$ for $n \times n$ matrix.
5. $\text{Adj}(AB) = \text{Adj}(B) \cdot \text{Adj}(A)$.

2.3.5 Inverse of a Matrix

If A is a square matrix and $|A| \neq 0$, the inverse of A is:

$$A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

Conditions:

- A must be square $(n \times n)$.
- $|A| \neq 0$ (non-singular matrix).

2.3.6 Properties of Inverse

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. kA inverse: $(kA)^{-1} = \frac{1}{k}A^{-1}$ (for $k \neq 0$)
5. Identity: $A \cdot A^{-1} = A^{-1} \cdot A = I$

3. Linearly Independent and Dependent vectors

3.1 Linearly Independent Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space is linearly independent if no vector in the set can be expressed as a linear combination of the others.

Mathematically:

Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$$

implies:

$$c_1 = c_2 = \dots = c_n = 0$$

3.2 Linearly Dependent Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly dependent if at least one vector can be written as a linear combination of the others.

Mathematically:

Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent if there exist scalars $c_1, c_2, \dots, c_n$, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$$

3.3 Test for Linear Dependence Using Determinants

If we arrange vectors as columns of a square matrix A :

- If $\det(A) \neq 0 \rightarrow$ vectors are linearly independent.
- If $\det(A) = 0 \rightarrow$ vectors are linearly dependent.

4. Rank

4.1 Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent rows or columns in the matrix. (OR) Rank = number of non-zero rows in Row Echelon Form (REF) or Reduced Row Echelon Form (RREF).

4.2 Rank using Minor

- The rank of a matrix is the largest order of any non-zero minor.

Steps:

1. Find the largest possible square sub-matrix.
2. Compute its determinant (minor).
3. If non-zero, rank = that order. If zero, try smaller order.

4.3 Properties of Rank

4. For zero matrix $0, \rho(0) = 0$
5. For identity matrix I_n of order $n, \rho(I_n) = n$
6. For $m \times n$ matrix $A, \rho(A) \leq \min(m, n)$
7. For scalar k and matrix A : if $k \neq 0, \rho(kA) = \rho(A)$; if $k = 0, \rho(kA) = 0$

8. For any matrix A, row rank = column rank = $\rho(A)$
9. For matrices A and B (product defined), $\rho(AB) \leq \min(\rho(A), \rho(B))$
10. For any matrix $A, \rho(A^T) = \rho(A)$
11. For matrices A and B of same order, $\rho(A + B) \leq \rho(A) + \rho(B)$
12. For invertible $n \times n$ matrix $A, \rho(A) = n$
13. For triangular matrix $A, \rho(A) =$ number of non-zero diagonal elements
14. System of Linear Equation

A system of linear equations is a set of equations where each equation is linear in the variables $x_1, x_2, \ldots, x_n$.

General Form:

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{gathered}
$$

Where:

- a_{ij} = coefficient of x_j in i^{th} equation
- b_i = constant term in i^{th} equation
- m = number of equations
- n = number of variables

The system can be written as:

$$AX = B$$

Where:

- A = coefficient matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$
- X = column vector of variables $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
- B = column vector of constants $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

5.1 Homogeneous System of Linear Equations

$$AX = 0$$

where A is an $m \times n$ coefficient matrix, X is $n \times 1$ variable vector, and the zero is $m \times 1$ zero vector.
Solutions:

1. Trivial Solution: $X = 0$ (all variables = 0)
2. Non-Trivial Solution: Any solution other than trivial. Exists if $\rho(A) < n$.

Properties:

- Always has at least the trivial solution.
- If $\rho(A) = n$, only trivial solution exists.
- If $\rho(A) < n$, infinite non-trivial solutions exist.
- Solutions form a vector subspace of $\mathbb{R}^n$ (solution space).

5.2 Non-Homogeneous System of Linear Equations

$$AX = B$$

where B is a non-zero $m \times 1$ column vector.
Solution Conditions (Rouché-Capelli Theorem):

- Let $\rho(A)$ = rank of coefficient matrix.
- Let $\rho([A \mid B])$ = rank of augmented matrix.

1. No Solution: If $\rho(A) \neq \rho([A \mid B])$.
2. Unique Solution: If $\rho(A) = \rho([A \mid B]) = n$.
3. Infinite Solutions: If $\rho(A) = \rho([A \mid B]) < n$.
4. LU Decomposition

Definition:
LU decomposition is the factorization of a square matrix A into the product of a Lower triangular matrix L and an Upper triangular matrix U :

$$A = LU$$

where:

- L = lower triangular matrix with 1 's on the diagonal ($l_{ii} = 1$)
- U = upper triangular matrix.

7. Eigen Values and Eigen vectors

Definition:
For a square matrix $A_{n \times n}$, a scalar λ is called an eigenvalue if there exists a non-zero vector X such that:

$$AX = \lambda X$$

For non-trivial solution ($X \neq 0$),

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation.
Steps to Find Eigenvalues and Eigenvectors:

1. Form $A - \lambda I$.

2. Compute $\det(A - \lambda I) = 0$ to find eigenvalues $\lambda_1, \lambda_2, \ldots$.
3. For each λ, solve $(A - \lambda I)X = 0$ to get the eigenvectors.

Properties:

1. Sum of eigenvalues = Trace of A.
2. Product of eigenvalues = Determinant of A.
3. If A is triangular, eigenvalues are its diagonal elements.
4. Eigenvalues of A^{-1} are reciprocals of eigenvalues of A.
5. Eigenvalues of kA are k times eigenvalues of A.
6. Diagonalizability of a Matrix

Definition:

A square matrix $A_{n \times n}$ is diagonalizable if there exists an invertible matrix P such that:

$$P^{-1}AP = D$$

where D is a diagonal matrix containing the eigenvalues of A.

Conditions for Diagonalizability:

1. A must have n linearly independent eigenvectors.
2. The sum of the geometric multiplicities of all distinct eigenvalues must be n.
3. If A has $\mathbf{n}$ distinct eigenvalues, it is always diagonalizable.

Properties:

1. A and D are similar matrices ($A \sim D$).
2. Powers of A are easy to compute: $A^k = PD^kP^{-1}$.
3. If A is symmetric, it is always diagonalizable with orthogonal P.
4. Diagonalizability does not require distinct eigenvalues, but requires enough independent eigenvectors.
5. Vector Space

9.1 Vector Space

Definition:

A vector space V over a field F is a set of elements (called vectors) along with two operations:

1. Vector Addition: $u + v \in V$ for all $u, v \in V$
2. Scalar Multiplication: $cu \in V$ for all $c \in F, u \in V$

Axioms of Vector Space:

(For all $u, v, w \in V$ and $a, b \in F$)

1. Closure under addition: $u + v \in V$
2. Commutativity: $u + v = v + u$
3. Associativity: $(u + v) + w = u + (v + w)$
4. Existence of additive identity: There exists $0 \in V$ such that $u + 0 = u$
5. Existence of additive inverse: For each u, there exists $(-u)$ such that $u + (-u) = 0$
6. Closure under scalar multiplication: $au \in V$
7. Distributive property over vector addition: $a(u + v) = au + av$
8. Distributive property over scalar addition: $(a + b)u = au + bu$
9. Associativity of scalar multiplication: $a(bu) = (ab)u$
10. Multiplicative identity: $1u = u$
    9.2 Subspace

Definition:

A subset $W \subseteq V$ is called a subspace of vector space V if:

1. W is non-empty (contains the zero vector).
2. Closed under addition: $u + v \in W$ for all $u, v \in W$.
3. Closed under scalar multiplication: $cu \in W$ for all $c \in F, u \in W$.

Important Notes:

- Every subspace is itself a vector space under the same operations as V.
- The intersection of subspaces is also a subspace.
- The sum of subspaces $U + W = \{u + w \mid u \in U, w \in W\}$ is a subspace.

10. Basis

10.1 Definition

- Linearly Dependent Vectors:

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V over a field F is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_k$ (not all zero) such that:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

- Linearly Independent Vectors:

The set is linearly independent if the only solution to the above equation is:

$$c_1 = c_2 = \cdots = c_k = 0$$

10.2 Using Rank to Determine
Dependence/Independence

1. Arrange the given vectors as columns of a matrix A.
2. Find $\text{Rank}(A)$.
3. If $\text{Rank}(A) =$ number of vectors, the set is linearly independent.
4. If $\text{Rank}(A) <$ number of vectors, the set is linearly dependent.
   10.3 Spanning Set

- Let V be a vector space over a field F.
- A set of vectors $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$ is called a spanning set of V if every vector in V can be expressed as a linear combination of vectors in S.

Mathematically:

$$V = \text{span}(S) = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k \mid c_i \in F\}$$

10.4 Basis
Definition

- A basis of a vector space V is a set of linearly independent vectors that spans V.
- That is, every vector in V can be expressed uniquely as a linear combination of basis vectors.

Mathematical Formulation

If $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V :

1. Span: $\text{span}(B) = V$
2. Linear Independence: No vector in B can be expressed as a linear combination of others.

Properties of Basis

1. All bases of a finite-dimensional vector space have the same number of elements (dimension).
2. If V has dimension n, then:
   - Any n linearly independent vectors in V form a basis.
   - Any n spanning vectors in V form a basis.

3. The number of vectors in a basis = Rank of the matrix formed by basis vectors.

10.5 Dimension of a Vector Space
Definition

- The dimension of a vector space V is the number of vectors in any basis of V.
- Denoted as: $\dim(V)$

Properties

1. All bases of V have the same number of elements → this number is the dimension.
2. If V is spanned by n vectors, then $\dim(V) \leq n$.
3. If $\dim(V) = n$, any set of more than n vectors is linearly dependent.
4. If $\dim(V) = n$, any set of fewer than n vectors cannot span V.
5. For subspace W of V: $\dim(W) \leq \dim(V)$.
6. Linear Transformation
   11.1 Linear Transformation

Definition

- A linear transformation $T: V \rightarrow W$ between two vector spaces is a mapping that satisfies:

1. Additivity: $T(u + v) = T(u) + T(v)$ for all $u, v \in V$
2. Homogeneity (Scalar Multiplication): $T(cu) = cT(u)$ for all $u \in V, c \in \mathbb{R}$ (or field F)

Key Points

- Preserves vector addition and scalar multiplication.
- The domain of T is V, and the codomain is W.
- Image: $\text{Im}(T) = \{T(v) \mid v \in V\}$
- Kernel: $\text{Ker}(T) = \{v \in V \mid T(v) = 0_W\}$

Matrix Representation

- Every linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix A such that:

$$T(x) = A \cdot x$$

11.2 Kernel and Image & Rank-Nullity Theorem

Kernel (Null Space)

- **Definition:**

$$\text{Ker}(T) = \{v \in V \mid T(v) = 0_W\}$$

- Contains all vectors mapped to the zero vector in W.
- Dimension of Ker(T) = Nullity of T.

Image (Range)

- Definition:

$$\text{Im}(T) = \{T(v) \mid v \in V\}$$

- Set of all vectors in W that are outputs of T.
- Dimension of Im(T) = Rank of T.

Rank-Nullity Theorem

For a linear transformation $T: V \to W$,

$$\dim(V) = \text{Rank}(T) + \text{Nullity}(T)$$

Properties

1. If $\text{Ker}(T) = \{0\}$, then T is one-to-one.
2. If $\text{Im}(T) = W$, then T is onto.
3. Rank is the dimension of the column space of the matrix representing T.

11.3 Null Space, Left Null Space, Row Space, Column Space of Matrix A

Let A be an $m \times n$ matrix.

1. **Null Space (N(A))**
- **Definition:**

$$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

- Set of all solutions to the homogeneous system $Ax = 0$.
- Dimension = Nullity of A.

2. **Left Null Space ($N(A^T)$)**
- **Definition:**

$$N(A^T) = \{y \in \mathbb{R}^m \mid A^T y = 0\}$$

- Orthogonal complement of Column Space.
- Dimension $= m - \text{Rank}(A)$.

3. **Row Space (R(A'))**
- **Definition**:

$$R(A^T) = \text{ Span of the rows of } A$$

- Lies in $\mathbb{R}^n$.
- Orthogonal complement of Null Space.
- Dimension = Rank of A.

4. **Column Space (R(A))**
- **Definition:**

$$R(A) = \text{ Span of the columns of } A$$

- Lies in $\mathbb{R}^m$.
- Orthogonal complement of Left Null Space.
- Dimension = Rank of A.

Key Relationships (Fundamental Theorem of Linear Algebra)

1. $\dim(N(A)) + \dim\left(R(A^T)\right) = n$
2. $\dim\left(N(A^T)\right) + \dim(R(A)) = m$
3. Row Space and Null Space are orthogonal complements in $\mathbb{R}^n$.
4. Column Space and Left Null Space are orthogonal complements in $\mathbb{R}^m$.
5. Quadratic Forms

12.1 Definition

- A quadratic form in n variables is an expression

$$Q(x) = x^T A x$$

where x is an $n \times 1$ column vector and A is an $n \times n$ symmetric matrix.

12.2 Properties of Quadratic Form

1. $Q(x)$ is homogeneous of degree 2 in x.
2. If A is symmetric, $Q(x)$ is uniquely defined by A.
3. $Q(x)$ is positive definite if $Q(x) > 0$ for all $x \neq 0$.
4. $Q(x)$ is positive semidefinite if $Q(x) \geq 0$ for all x.
5. $Q(x)$ is negative definite if $Q(x) < 0$ for all $x \neq 0$.
6. $Q(x)$ is negative semidefinite if $Q(x) \leq 0$ for all x.
7. $Q(x)$ is indefinite if it takes both positive and negative values.

12.3 Classification of Quadratic Form using Eigenvalues

Let A be the symmetric matrix in $Q(x) = x^T A x$.

- If all eigenvalues $> 0 \to$ Positive Definite

- If all eigenvalues ≥ 0 (and at least one $= 0$) → Positive Semidefinite
- If all eigenvalues < 0 → Negative Definite
- If all eigenvalues ≤ 0 (and at least one $= 0$) → Negative Semidefinite
- If eigenvalues are of mixed signs → Indefinite

12.4 Classification of Quadratic Form using Leading Principal Minors

Let Δ_k = Determinant of the $k \times k$ leading principal submatrix of A.

- Positive Definite → All $\Delta_k > 0$ for $k = 1,2, \ldots, n$
- Negative Definite → $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \ldots$ (alternating signs)
- Positive Semidefinite → All $\Delta_k \geq 0$ and some $\Delta_k = 0$

13. Inner Product

Definition:

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An inner product on V is a function

$$\langle\cdot,\cdot\rangle: V \times V \to \mathbb{R} \text{ (or) } \mathbb{C}$$

that assigns to each pair of vectors $u, v \in V$ a scalar $\langle u, v\rangle$, satisfying:

13.1 Properties of Inner Product

1. Conjugate Symmetry: $\langle u, v\rangle = \overline{\langle v, u\rangle}$
2. Linearity in First Argument: $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$
3. Positive-Definiteness: $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \Leftrightarrow v = 0$

- Euclidean Inner Product (on $\mathbb{R}^n$):

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$$

- Complex Inner Product (on $\mathbb{C}^n$):

$$\langle u, v\rangle = \sum_{i=1}^{n} u_i \overline{v_i}$$

Norm Induced by Inner Product

$$\|v\| = \sqrt{\langle v, v\rangle}$$

13.2 Orthogonality & Orthonormal Sets

Orthogonality:

Two vectors $u, v \in V$ are orthogonal if

$$\langle u, v\rangle = 0$$

Orthonormal Set:

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in V is orthonormal if:

1. $\langle v_i, v_j\rangle = 0$ for $i \neq j$ (orthogonal)
2. $\|v_i\| = 1$ for all i (unit vectors)

Properties:

1. Orthonormal vectors are automatically linearly independent.
2. Any vector u in the span of an orthonormal set can be expressed as:

$$u = \sum_{i=1}^{k} \langle u, v_i\rangle v_i$$

3. Orthogonal sets simplify calculations of projections and norms.
4. Partitioned Matrix and its Properties

Definition:

A partitioned matrix (or block matrix) is a matrix divided into smaller submatrices (blocks), usually for ease of computation.

Example:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{ij} are submatrices.

Properties

1. Addition: If A and B have same partitioning, then $A + B = [A_{ij} + B_{ij}]$.
2. Scalar multiplication: $kA = [kA_{ij}]$.
3. Transpose: A^T is obtained by transposing each block and swapping their positions.
4. Multiplication: If partitions are conformable, $(AB)_{ij} = \sum_k A_{ik} B_{kj}$.
5. Inverse (2×2 block matrix): If A_{11} and Schur complements are invertible, block inversion formula applies.
6. Determinant (2×2 block matrix): If $A_{21} = 0, \det(A) = \det(A_{11}) \cdot \det(A_{22})$.
7. Block diagonal matrix: $\det(A) = \prod \det(A_{ii})$ and $A^{-1} = \operatorname{diag}(A_{11}^{-1}, A_{22}^{-1}, \ldots)$ if each A_{ii} is invertible.

Note:

For a block matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Assume A_{11} and the Schur complement $S = A_{22} - A_{21} A_{11}^{-1} A_{12}$ are invertible.

Formula:

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}$$

15. Singular Value Decomposition (SVD)

Definition:

For any real $m \times n$ matrix A, there exist orthogonal matrices $U(m \times m)$ and $V(n \times n)$, and a diagonal matrix $\Sigma(m \times n)$ with non-negative entries, such that:

$$A = U\Sigma V^T$$

where:

- U = left singular vectors (columns orthonormal)
- Σ = diagonal entries = singular values $(\sigma_1 \geq \sigma_2 \geq \cdots \geq 0)$
- V = right singular vectors (columns orthonormal)

Properties:

1. Singular values = square roots of eigenvalues of $A^T A$ or AA^T.
2. U diagonalizes AA^T, V diagonalizes $A^T A$.
3. Rank of A = number of non-zero singular values.
4. Best low-rank approximation given by first k largest singular values.
5. Works for all $m \times n$ matrices (square or rectangular).

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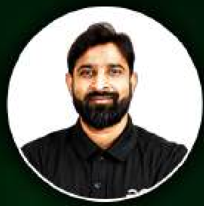

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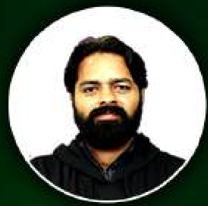

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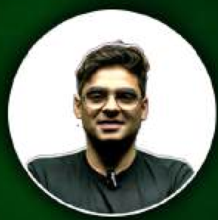

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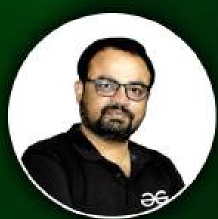

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