GeeksforGeeks

# GATE फर्रे

## DATA SCIENCE & ARTIFIACL INTELLIGENCE (DA)

## ENGINEERING MATHEMATICS

## SHORT NOTES

ENROLL NOW

TO EXCEL IN GATE
AND ACHIEVE YOUR DREAM IIT OR PSU!

ENROLL NOW

# Linear Algebra

## 1. Matrix

### 1.1 Matrices

A matrix is a rectangular arrangement of numbers, symbols, or expressions in rows and columns. It is generally denoted by capital letters (e.g., $A, B, C$ ) and represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Here:

- $a_{ij}$ = element in the $i^{\text{th}}$ row and $j^{\text{th}}$ column.
- $m$ = number of rows.
- $n$ = number of columns.

### 1.2 Order of a Matrix

The order of a matrix refers to its dimensions:

$$\text{Order } = m \times n$$

where:

- $m$ = number of rows
- $n$ = number of columns

### 1.3 Operations on Matrices

#### 1.3.1 Equality of Matrices

Two matrices $A$ and $B$ are equal if:

- They have the same order.
- Corresponding elements are equal: $a_{ij} = b_{ij}$.

#### 1.3.2 Addition

If $A$ and $B$ are of same order:

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

#### 1.3.3 Subtraction

If $A$ and $B$ are of same order:

$$(A - B)_{ij} = a_{ij} - b_{ij}$$

#### 1.3.4 Scalar Multiplication

If $k$ is a scalar:

$$(kA)_{ij} = k \cdot a_{ij}$$

#### 1.3.5 Matrix Multiplication

If $A$ is $m \times n$ and $B$ is $n \times p$.

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

### 1.4 Transpose of a Matrix

The transpose of a matrix $A$, denoted $A^T$, is obtained by interchanging rows and columns:

$$(A^T)_{ij} = a_{ji}$$

Let $A$ and $B$ be matrices of appropriate order, and $k$ be a scalar.

1. Double Transpose
$$(A^T)^T = A$$
2. Transpose of a Sum
$$(A + B)^T = A^T + B^T$$
3. Transpose of a Difference
$$(A - B)^T = A^T - B^T$$
4. Transpose of a Scalar Multiple
$$(kA)^T = kA^T$$
5. Transpose of a Product
$$(AB)^T = B^T A^T$$
6. Transpose of an Inverse (if $A$ is invertible)
$$(A^{-1})^T = (A^T)^{-1}$$

### 1.5 Trace of a Matrix

The trace of a square matrix $A$ is the sum of its diagonal elements:

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Let $A$ and $B$ be square matrices of the same order, and $k$ be a scalar.

1. Trace of a Scalar Multiple
$$\text{tr}(kA) = k \cdot \text{tr}(A)$$
2. Trace of a Sum
$$\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$$
3. Trace of a Difference
$$\text{tr}(A - B) = \text{tr}(A) - \text{tr}(B)$$
4. Trace of a Product (Commutative Property for Trace)
$$\text{tr}(AB) = \text{tr}(BA)$$

**Note**: This holds even if $AB \neq BA$.)

5. Cyclic Property for Multiple Matrices
$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB)$$
6. Trace of a Transpose
$$\text{tr}(A^T) = \text{tr}(A)$$

### 1.6 Types of Matrices

**1.6.1 Row Matrix -** Only one row.

**1.6.2 Column Matrix -** Only one column.

**1.6.3 Square Matrix - Number of rows =** number of columns.

**1.6.4 Zero/Null Matrix -** All elements are zero.

**1.6.5 Identity Matrix -** Square matrix with 1's on main diagonal and 0's elsewhere.

**1.6.6 Diagonal Matrix -** Non-zero elements only on the main diagonal.

**1.6.7 Scalar Matrix -** Diagonal matrix with equal diagonal elements.

**1.6.8 Upper Triangular Matrix -** All elements below main diagonal are zero.

**1.6.9 Lower Triangular Matrix -** All elements above main diagonal are zero.

**1.6.10 Symmetric Matrix-** $A^T = A$.

**1.6.11 Skew-Symmetric Matrix** $-A^T = -A$.

**1.6.12 Orthogonal Matrix** $-A^T A = I$.

**1.6.13 Hermitian Matrix -** $A^H = A$ (complex conjugate transpose).

**1.6.14 Skew-Hermitian Matrix-** $A^H = -A$.

**1.6.15 Involutory Matrix** $-A^2 = I$.

**1.6.16 Idempotent Matrix -** A matrix $A$ is idempotent if $A^2 = A$.

**1.6.17 Nilpotent Matrix -** A matrix $A$ is nilpotent if $A^k = 0$ for some positive integer $k$.

## 2. Determinant and its properties

### 2.1 Determinant

A determinant is a scalar value calculated from a square matrix.
It is denoted as $\det(A)$ or $|A|$.

For a $2 \times 2$ matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\det(A) = ad - bc$$

For a $3 \times 3$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

**Geometrical Meaning:**

- In 2D, determinant represents the area of the parallelogram formed by column vectors.
- In 3D, determinant represents the volume of the parallelepiped formed by column vectors.

### 2.2 Properties of Determinants

1. For $n \times n$ matrix $A, \det(A^T) = \det(A)$
2. If two rows or columns of $A$ are interchanged, $\det(A)$ changes sign
3. If two rows or columns of $A$ are identical, $\det(A) = 0$
4. If a row or column of $A$ is multiplied by $k$, determinant is multiplied by $k$
5. A common factor from any row or column of $A$ can be taken outside $\det(A)$
6. If any row or column of $A$ is all zeros, $\det(A) = 0$
7. Adding a multiple of one row/column to another in $A$ does not change $\det(A)$
8. For $n \times n$ matrices $A$ and $B, \det(AB) = \det(A) \cdot \det(B)$
9. For invertible $n \times n$ matrix $A, \det(A^{-1}) = \frac{1}{\det(A)}$
10. For triangular $n \times n$ matrix $A, \det(A) =$ product of diagonal elements

### 2.3 Adjoint and Inverse of a Matrix

#### 2.3.1 Minor of an Element

For a square matrix $A = [a_{ij}]$, the minor of an element $a_{ij}$ is the determinant of the submatrix obtained by deleting the $i^{th}$ row and $j^{th}$ column from $A$.

$M_{ij}$ = determinant of submatrix after removing row $i$ and column $j$.

#### 2.3.2 Cofactor of an Element

The cofactor of $a_{ij}$ is:

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

Where:

- $(-1)^{i+j}$ introduces the alternating sign pattern.

#### 2.3.3 Adjoint (Adjugate) of a Matrix

The adjoint (or adjugate) of $A$ is the transpose of the cofactor matrix.

If:

$$\text{Cofactor Matrix of } A = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

Then:

$$\text{Adj}(A) = [C_{ij}]^T$$

#### 2.3.4 Properties of Adjoint

1. $A \cdot \text{Adj}(A) = \text{Adj}(A) \cdot A = |A| \cdot I_n$
2. $\text{Adj}(A^T) = (\text{Adj}(A))^T$
3. If $A$ is invertible, $\text{Adj}(A) \neq 0$.
4. $\text{Adj}(kA) = k^{n-1} \cdot \text{Adj}(A)$ for $n \times n$ matrix.
5. $\text{Adj}(AB) = \text{Adj}(B) \cdot \text{Adj}(A)$.

#### 2.3.5 Inverse of a Matrix

If $A$ is a square matrix and $|A| \neq 0$, the inverse of $A$ is:

$$A^{-1} = \frac{\text{Adj}(A)}{|A|}$$

**Conditions:**

- $A$ must be square ($n \times n$).
- $|A| \neq 0$ (non-singular matrix).

#### 2.3.6 Properties of Inverse

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $kA$ inverse: $(kA)^{-1} = \frac{1}{k}A^{-1}$ (for $k \neq 0$ )
5. Identity: $A \cdot A^{-1} = A^{-1} \cdot A = I$

### 3. Linearly Independent and Dependent vectors

#### 3.1 Linearly Independent Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space is linearly independent if no vector in the set can be expressed as a linear combination of the others.

**Mathematically:**

Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$$

implies:

$$c_1 = c_2 = \dots = c_n = 0$$

#### 3.2 Linearly Dependent Vectors

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly dependent if at least one vector can be written as a linear combination of the others.

**Mathematically:**

Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly dependent if there exist scalars $c_1, c_2, \dots, c_n$, not all zero, such that:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$$

#### 3.3 Test for Linear Dependence Using Determinants

If we arrange vectors as columns of a square matrix $A$ :

- If $\det(A) \neq 0 \rightarrow$ vectors are linearly independent.
- If $\det(A) = 0 \rightarrow$ vectors are linearly dependent.

### 4. Rank

#### 4.1 Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent rows or columns in the matrix.

(OR) Rank = number of non-zero rows in Row Echelon Form (REF) or Reduced Row Echelon Form (RREF).

### 4.2 Rank using Minor

- The rank of a matrix is the largest order of any non-zero minor.

**Steps:**

1. Find the largest possible square sub-matrix.
2. Compute its determinant (minor).
3. If non-zero, rank = that order. If zero, try smaller order.

### 4.3 Properties of Rank

1. For zero matrix $0, \rho(0) = 0$
2. For identity matrix $I_n$ of order $n, \rho(I_n) = n$
3. For $m \times n$ matrix $A, \rho(A) \leq \min(m, n)$
4. For scalar $k$ and matrix $A$ : if $k \neq 0, \rho(kA) = \rho(A)$; if $k = 0, \rho(kA) = 0$
5. For any matrix $A$, row rank = column rank = $\rho(A)$
6. For matrices $A$ and $B$ (product defined), $\rho(AB) \leq \min(\rho(A), \rho(B))$
7. For any matrix $A, \rho(A^T) = \rho(A)$
8. For matrices $A$ and $B$ of same order, $\rho(A + B) \leq \rho(A) + \rho(B)$
9. For invertible $n \times n$ matrix $A, \rho(A) = n$
10. For triangular matrix $A, \rho(A)$ = number of non-zero diagonal elements

## 5. System of Linear Equation

A system of linear equations is a set of equations where each equation is linear in the variables $x_1, x_2, \dots, x_n$.

**General Form:**

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

**Where:**

- $a_{ij}$ = coefficient of $x_j$ in $i^{\text{th}}$ equation
- $b_i$ = constant term in $i^{\text{th}}$ equation
- $m$ = number of equations
- $n$ = number of variables

The system can be written as:

$$AX = B$$

Where:

- $A$ = coefficient matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$
- $X$ = column vector of variables $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$
- $B$ = column vector of constants $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

### 5.1 Homogeneous System of Linear Equations

$$AX = 0$$

where $A$ is an $m \times n$ coefficient matrix, $X$ is $n \times 1$ variable vector, and the zero is $m \times 1$ zero vector.
Solutions:

1. Trivial Solution: $X = 0$ (all variables = 0)
2. Non-Trivial Solution: Any solution other than trivial. Exists if $\rho(A) < n$.

**Properties:**

- Always has at least the trivial solution.
- If $\rho(A) = n$, only trivial solution exists.
- If $\rho(A) < n$, infinite non-trivial solutions exist.
- Solutions form a vector subspace of $\mathbb{R}^n$ (solution space).

### 5.2 Non-Homogeneous System of Linear Equations

$$AX = B$$

where $B$ is a non-zero $m \times 1$ column vector.

**Solution:**

Conditions (Rouché-Capelli Theorem):

- Let $\rho(A)$ = rank of coefficient matrix.
- Let $\rho([A \mid B])$ = rank of augmented matrix.

1. No Solution: If $\rho(A) \neq \rho([A \mid B])$.
2. Unique Solution: If $\rho(A) = \rho([A \mid B]) = n$.
3. Infinite Solutions: If $\rho(A) = \rho([A \mid B]) < n$.

### 6. LU Decomposition

**Definition:**

LU decomposition is the factorization of a square matrix $A$ into the product of a Lower triangular matrix $L$ and an Upper triangular matrix $U$ :

$$A = LU$$

where:

- $L =$ lower triangular matrix with 1 's on the diagonal ( $l_{ii} = 1$ )
- $U =$ upper triangular matrix.

### 7. Eigen Values and Eigen vectors

**Definition:**

For a square matrix $A_{n\times n}$, a scalar $\lambda$ is called an eigenvalue if there exists a non-zero vector $X$ such that:

$$AX = \lambda X$$

For non-trivial solution ( $X \neq 0$ ),

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation.

Steps to Find Eigenvalues and Eigenvectors:

1. Form $A - \lambda I$.
2. Compute $\det(A - \lambda I) = 0$ to find eigenvalues $\lambda_1, \lambda_2, \ldots$.
3. For each $\lambda$, solve $(A - \lambda I)X = 0$ to get the eigenvectors.

**Properties:**

1. Sum of eigenvalues = Trace of $A$.
2. Product of eigenvalues = Determinant of $A$.
3. If $A$ is triangular, eigenvalues are its diagonal elements.
4. Eigenvalues of $A^{-1}$ are reciprocals of eigenvalues of $A$.
5. Eigenvalues of $kA$ are $k$ times eigenvalues of $A$.

### 8. Diagonalizability of a Matrix

**Definition:**

A square matrix $A_{n\times n}$ is diagonalizable if there exists an invertible matrix $P$ such that:

$$P^{-1}AP = D$$

where $D$ is a diagonal matrix containing the eigenvalues of $A$.

**Conditions for Diagonalizability:**

1. $A$ must have n linearly independent eigenvectors.
2. The sum of the geometric multiplicities of all distinct eigenvalues must be $n$.
3. If $A$ has $\mathbf{n}$ distinct eigenvalues, it is always diagonalizable.

**Properties:**

1. $A$ and $D$ are similar matrices ($A \sim D$).
2. Powers of $A$ are easy to compute: $A^k = PD^kP^{-1}$.
3. If $A$ is symmetric, it is always diagonalizable with orthogonal $P$.
4. Diagonalizability does not require distinct eigenvalues, but requires enough independent eigenvectors.

### 9 Null Space, Left Null Space, Row Space, Column Space of Matrix $A$

Let $A$ be an $m \times n$ matrix.

1. **Null Space (N(A))**
   - **Definition:**
     $$N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$$
   - Set of all solutions to the homogeneous system $Ax = 0$.
   - Dimension = Nullity of $A$.
2. **Left Null Space ( $N(A^\top)$ )**
   - **Definition:**
     $$N(A^T) = \{y \in \mathbb{R}^m \mid A^T y = 0\}$$
   - Orthogonal complement of Column Space.
   - Dimension $= m - \text{Rank}(A)$.
3. **Row Space (R(A'))**
   - **Definition**:
     $$R(A^T) = \text{ Span of the rows of } A$$
   - Lies in $\mathbb{R}^n$.
   - Orthogonal complement of Null Space.
   - Dimension = Rank of $A$.
4. **Column Space (R(A))**
   - **Definition:**
     $$R(A) = \text{ Span of the columns of } A$$
   - Lies in $\mathbb{R}^m$.

- Orthogonal complement of Left Null Space.
- Dimension = Rank of $A$.

**Key Relationships (Fundamental Theorem of Linear Algebra)**

1. $\dim(N(A)) + \dim\left(R(A^T)\right) = n$
2. $\dim\left(N(A^T)\right) + \dim(R(A)) = m$
3. Row Space and Null Space are orthogonal complements in $\mathbb{R}^n$.
4. Column Space and Left Null Space are orthogonal complements in $\mathbb{R}^m$.

# Probability & Statistics GATE DA

## 1. Counting

### 1.1 Basic Counting Principles

**THE PRODUCT RULE** Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

**THE SUM RULE** If a task can be done either in one of $n_1$ ways or in one of $n_2$ ways, where none of the set of $n_1$ ways is the same as any of the set of $n_2$ ways, then there are $n_1 + n_2$ ways to do the task.

**Principle of inclusion–exclusion**

If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
Suppose that A1 and A2 are sets.
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

### 1.2 Pigeon Hole Principle

**THE PIGEONHOLE PRINCIPLE** If $k$ is a positive integer and $k + 1$ or more objects are placed into $k$ boxes, then there is at least one box containing two or more of the objects.
**THE GENERALIZED PIGEONHOLE PRINCIPLE** If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

### 1.3 Permutation

A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.
The number of r-permutations of a set with n elements is denoted by P(n, r).
If $n$ and $r$ are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$.

### 1.4 Combinations

An r-combination of elements of a set is an unordered selection of r elements from the set.
The number of $r$-combinations of a set with $n$ elements, where $n$ is a nonnegative integer and $r$ is an integer with $0 \leq r \leq n$, equals
$C(n, r) = \frac{n!}{r!(n-r)!}$

### 1.5 Binomial Theorem

THE BINOMIAL THEOREM Let $x$ and $y$ be variables, and let $n$ be a nonnegative integer. Then
$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$.

**Note:**

1. Let $n$ and $r$ be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.
2. PASCAL'S IDENTITY Let $n$ and $k$ be positive integers with $n \geq k$. Then
$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

### 1.6 Generalized Permutations and Combinations

| Type | Condition | Formula | Explanation |
|---|---|---|---|
| Permutation (Without Repetition) | Order matters, no repetition | $P(n,r) = \frac{n!}{(n-r)!}$ | Selecting and arranging $r$ items from $n$ distinct items |
| Permutation (With Repetition) | Order matters, repetition allowed | $n^r$ | Selecting $r$ items from $n$ options, each can repeat |
| Combination (Without Repetition) | Order doesn't matter, no repetition | $C(n,r) = \frac{n!}{r!\,(n-r)!}$ | Selecting $r$ items from $n$ distinct items, order doesn't matter |
| Combination (With Repetition) | Order doesn't matter, repetition allowed | $C(n+r-1,r) = \frac{(n+r-1)!}{r!\,(n-1)!}$ | Selecting $r$ items from $n$ types, repetitions allowed, order irrelevant |

**Note:**

The number of ways to distribute $n$ distinguishable objects into $k$ distinguishable boxes so that $n_i$ objects are placed into box $i, i = 1,2,\dots,k$, equals $\frac{n!}{n_1!n_2!\cdots n_k!}$

### 1.7 Distribution of r balls into n boxes

| | $n$ **distinguishable boxes** | | $n$ **indistinguishable boxes** | |
|---|---|---|---|---|
| | empty box allowed | no box empty | empty box allowed | no box empty |
| r disting. balls | $n^r$ | $n!\left\{{r \atop n}\right\}$ | $\sum_{i=1}^{n}\left\{{r \atop i}\right\}$ | $\left\{{r \atop n}\right\}$ |
| r indisting. balls | $\binom{r+n-1}{r}$ | $\binom{r-1}{n-1}$ | $\sum_{i=1}^{n}\left\lvert{r \atop i}\right\rvert$ | $\left\lvert{r \atop n}\right\rvert$ |

- $\left\lvert{r \atop n}\right\rvert$ : Number of integer partitions of $r$ into $n$ parts (used for indistinguishable balls & boxes).
- $\binom{a}{b}$ : Binomial coefficient.
- $S(r,i) = \left\{{r \atop i}\right\} = \frac{1}{i!}\sum_{k=0}^{i} (-1)^k\binom{i}{k}(i-k)^r$

## 2. Axioms of Probability

### 2.1 Sample Space and Events

Consider an experiment whose outcome will not be known in advance, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the sample space of the experiment and is denoted by S.

Any subset E of the sample space is known as an event

### 2.2 Probability

Probability refers to the extent of occurrence of events.

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of equally likely outcomes}} = \frac{n(E)}{n(S)}$$

**Where:**

- P(E): Probability of event E
- n(E): Number of favourable outcomes for event E
- n(S): Total number of outcomes in the sample space

Note: Let A, E, F are the events associated with a random experiment, then

(i) Probability of non occurrence of event A, i.e, P(A') = 1 – P(A)

(ii)If E is a subset of F, then P(E) <= P(F)

(iii) P(E ∪ F) = P(E) + P(F) − P(E ∩ F)

**The three axioms of probability**

**Axiom 1**

$$0 \le P(E) \le 1$$

**Axiom 2**

$$P(S) = 1$$

**Axiom 3**

For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_iE_j = \emptyset$ when $i \neq j$ ),

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

We refer to $P(E)$ as the probability of the event $E$.

### 2.3 Types of Events

**1. Sure (Certain) Event:**

An event that is guaranteed to happen.
Formula: $P(E) = 1$

**2. Impossible Event:**

An event that cannot occur under any circumstances.
Formula: $P(E) = 0$

**3. Simple (Elementary) Event:**

An event that consists of only a single outcome.

**4. Compound (Composite) Event:**

An event that consists of two or more outcomes.

**5. Mutually Exclusive Events:**

Two events are mutually exclusive if they cannot occur at the same time.
Formula: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = 0$

**6. Exhaustive Events:**

A set of events is exhaustive if at least one of them must occur.
Formula: $P(E_1) + P(E_2) + \cdots + P(E_n) = 1$

**7. Independent Events:**

Events are independent if the occurrence of one does not affect the occurrence of the other.
Formula: $P(A \cap B) = P(A) \cdot P(B)$

**8. Dependent Events:**

Events are dependent if the occurrence of one affects the probability of the other.
Formula: $P(A \cap B) = P(A) \cdot P(B \mid A)$

**9. Complementary Events:**

Two events are complementary if one occurs exactly when the other does not.
Formula: $P(E') = 1 - P(E)$

## 3. Conditional Probability & Baye's Theorem

### 3.1 Conditional Probability

The conditional probability of an event $A$ given that another event $B$ has occurred is the probability of $A$ occurring under the condition that $B$ has already occurred.

**Formula**

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) > 0$$

- $P(A \mid B)$ : Probability of $A$ given $B$
- $P(A \cap B)$ : Probability that both $A$ and $B$ occur
- $P(B)$ : Probability that $B$ occurs

**Multiplication Rule of Probability**

$$P(A \cap B) = P(A \mid B) \cdot P(B) = P(B \mid A) \cdot P(A)$$

### 3.2 Law of Total Probability

If $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events (i.e., one of them must occur), and $A$ is any event, then:

$$P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i)$$

- Useful when the probability of $A$ depends on different cases $B_1, B_2, \ldots, B_n$
- Requires: $B_1 \cup B_2 \cup \cdots \cup B_n = S$ and $B_i \cap B_j = \emptyset$ for $i \neq j$

### 3.2 Baye's Theorem

$P(B \mid A)$ : Probability of event $B$ given $A$ occurred

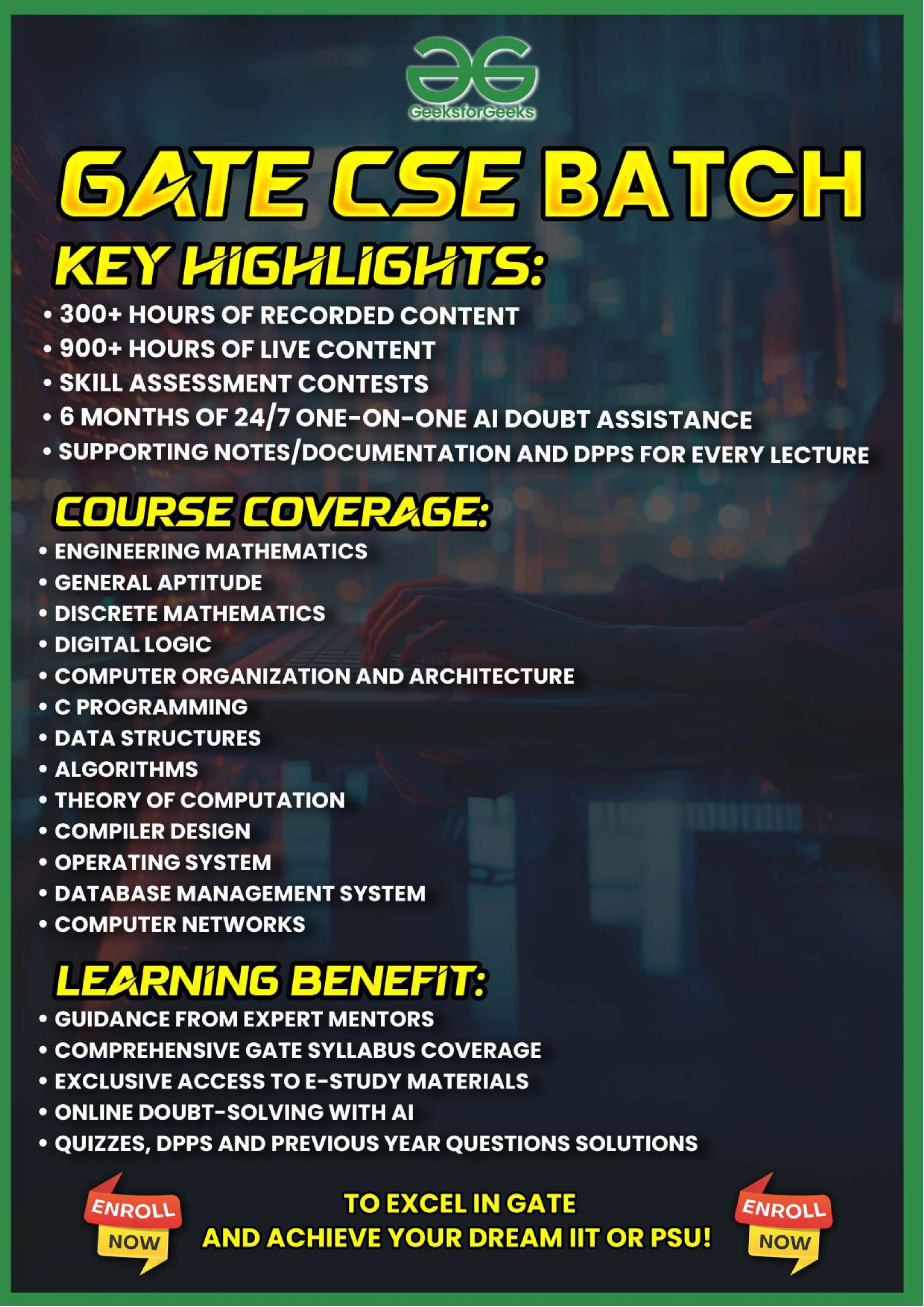
$$P(B \mid A) = \frac{P(B) \cdot P(A \mid B)}{P(A)}$$

### 3.4 Generalized Baye's theorem

Used to reverse conditional probabilities: Find $P(B_i \mid A)$ when $P(A \mid B_i)$ is known.

$$P(B_i \mid A) = \frac{P(B_i) \cdot P(A|B_i)}{\sum_{j=1}^{n} P(B_j) \cdot P(A|B_j)}$$

- $B_1, B_2, \ldots, B_n$ : Partition of the sample space
- $A$ : Observed evidence/event
- Denominator is from Law of Total Probability


GeeksforGeeks
GATE CSE BATCH
KEY HIGHLIGHTS:
• 300+ HOURS OF RECORDED CONTENT
• 900+ HOURS OF LIVE CONTENT
• SKILL ASSESSMENT CONTESTS
• 6 MONTHS OF 24/7 ONE-ON-ONE AI DOUBT ASSISTANCE
• SUPPORTING NOTES/DOCUMENTATION AND DPPS FOR EVERY LECTURE
COURSE COVERAGE:
• ENGINEERING MATHEMATICS
• GENERAL APTITUDE
• DISCRETE MATHEMATICS
• DIGITAL LOGIC
• COMPUTER ORGANIZATION AND ARCHITECTURE
• C PROGRAMMING
• DATA STRUCTURES
• ALGORITHMS
• THEORY OF COMPUTATION
• COMPILER DESIGN
• OPERATING SYSTEM
• DATABASE MANAGEMENT SYSTEM
• COMPUTER NETWORKS
LEARNING BENEFIT:
• GUIDANCE FROM EXPERT MENTORS
• COMPREHENSIVE GATE SYLLABUS COVERAGE
• EXCLUSIVE ACCESS TO E-STUDY MATERIALS
• ONLINE DOUBT-SOLVING WITH AI
• QUIZZES, DPPS AND PREVIOUS YEAR QUESTIONS SOLUTIONS
ENROLL NOW
TO EXCEL IN GATE
AND ACHIEVE YOUR DREAM IIT OR PSU!
ENROLL NOW

## 4. Key measures of central tendency and dispersion

### 4.1 Mean

The average value of a data set.

**Formula:**

$$\text{Mean } = \bar{x} = \frac{\sum x_i}{n}$$

Where $x_i$ are the data values and $n$ is the total number of values.

### 4.2 Median

The middle value when the data is arranged in ascending or descending order.

- If $n$ is odd: median is the middle term
- If $n$ is even: median is the average of the two middle terms (No formula needed; depends on sorting)

### 4.3 Mode

The value(s) that occur most frequently in the dataset.

- A dataset may be unimodal, bimodal, or multimodal

### 4.4 Variance

A measure of how much the data values deviate from the mean.

Formula (Population):

$$\sigma^2 = \frac{\sum (x_i - \mu)^2}{N}$$

**Formula (Sample):**

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

### 4.5 Standard Deviation

The square root of the variance; gives spread in the same units as the data.

Formula:

$$\sigma = \sqrt{\sigma^2} \text{ or } s = \sqrt{s^2}$$

## 5. Random Variable

### 5.1 Random Variable - Definition

A random variable is a function that assigns a real number to each outcome of a random experiment.

### 5.2 Types of Random Variables

- **Discrete Random Variable:**
  Takes countable values (finite or countably infinite)
  Example: Number of heads in 3 coin tosses

- **Continuous Random Variable:**
  Takes uncountably infinite values (within intervals)
  Example: Temperature in a day

### 5.3 Probability Distribution Function (PDF / PMF)

**A. Discrete Random Variable (PMF - Probability Mass Function)**

For a discrete random variable $X$ :

$$P(X = x_i) = p_i,$$

where

- $x_i$ : possible value that $X$ can take
- $p_i = P(X = x_i)$ : probability that $X$ takes the value $x_i$

$$\sum_i p_i = 1 \text{ and } 0 \le p_i \le 1$$

**B. Continuous Random Variable (PDF - Probability Density Function)**

For a continuous random variable $X$ :

- $f(x)$ : probability density function, such that

$$P(a \le X \le b) = \int_a^b f(x)dx$$

Conditions:

$$f(x) \ge 0 \text{ for all } x, \text{ and } \int_{-\infty}^{\infty} f(x)dx = 1$$

### 5.4 Cumulative Distribution Function (CDF)

Gives the probability that $X$ takes a value less than or equal to $x$ :

- For discrete:

$$F(x) = \sum_{x_i \le x} P(X = x_i)$$

- For continuous:

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

**Relationship between PDF and CDF (Continuous Case)**

If $F(x)$ is the Cumulative Distribution Function (CDF) of a continuous random variable $X$, then:

$$f(x) = \frac{d}{dx}F(x)$$

That is, the PDF is the derivative of the CDF.

### 5.5 Expected Value (Mean)

- For Discrete:

$$E(X) = \sum x_i \cdot P(x_i)$$

- For Continuous:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

### 5.6 Variance and Standard Deviation

- **Variance:**

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

- **Standard Deviation:**

$$\sigma = \sqrt{\text{Var}(X)}$$

### 5.7 Properties of Expectation

Let $X$ and $Y$ be random variables and $a, b \in \mathbb{R}$ :

1. **Linearity:**

$$E(aX + b) = aE(X) + b$$

2. **Sum Rule:**

$$E(X + Y) = E(X) + E(Y)$$

3. **If X is constant $c$ :**

$$E(c) = c$$

4. If $X$ and $Y$ are independent random variables, $E(XY) = E(X)E(Y)$ I

### 5.8 Properties of Variance

1. **Scaling and Shifting:**

$$\text{Var}(aX + b) = a^2 \cdot \text{Var}(X)$$

2. **If $X$ and $Y$ are independent:**

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

### 5.9 Conditional Distribution

**(a) Discrete Case - Conditional PMF**

For discrete random variables $X$ and $Y$, the conditional probability mass function of $X$ given $Y = y$ is:

$$P_{X|Y}(x \mid y) = \frac{P(X = x, Y = y)}{P(Y = y)} \quad \text{if } P(Y = y) > 0$$

**(b) Continuous Case - Conditional PDF**

For continuous random variables $X$ and $Y$, the conditional probability density function of $X$ given $Y = y$ is:

$$f_{XYY}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} \quad \text{if } f_Y(y) > 0$$

### 5.10 Conditional Expectation

Conditional expectation gives the expected value of a random variable given that another variable has a specific value.

**(a) Discrete Case**

If $X$ and $Y$ are discrete random variables, the conditional expectation of $X$ given $Y = y$ is:

$$E[X \mid Y = y] = \sum_{z} x \cdot P(X = x \mid Y = y)$$

**(b) Continuous Case**

If $X$ and $Y$ are continuous random variables, the conditional expectation of $X$ given $Y = y$ is:

$$E[X \mid Y = y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x \mid y)dx$$

Here, $f_{XY}(x \mid y)$ is the conditional PDF of $X$ given $Y = y$.

### 5.11 Law of Total Expectation

**(a) Discrete Case**

$$E[X] = \sum_{y} E[X \mid Y = y] \cdot P(Y = y)$$

**(b) Continuous Case**

$$E[X] = \int_{-\infty}^{\infty} E[X \mid Y = y] \cdot f_Y(y)dy$$

## 6. Types of Random Variable

### 6.1 Discrete Distributions

1. **Discrete Uniform Distribution**

- **Notation**: $X \sim U(a, b)$
- **Definition**: Takes integer values from $a$ to $b$ with equal probability.
- **PMF**: $P(X = x) = \frac{1}{b-a+1}$, for $x \in \{a, a+1, \dots, b\}$
- **CDF**: $F(x) = \frac{|x|-a+1}{b-a+1}$, for $x \geq a$
- **Expectation**: $E[X] = \frac{a+b}{2}$
- **Variance**: $\mathrm{Var}(X) = \frac{(b-a+1)^2-1}{12}$
- **Standard Deviation**: $\sqrt{\mathrm{Var}(X)}$

2. **Bernoulli Distribution**

- **Notation**: $X \sim \mathrm{Bern}(p)$
- **Definition**: Random variable with two outcomes: success (1) and failure (0).
- **PMF**: $P(X = x) = p^x(1-p)^{1-x}$, for $x \in \{0,1\}$
- **CDF**: Step function at 0 and 1
- **Expectation**: $E[X] = p$
- **Variance**: $\mathrm{Var}(X) = p(1-p)$
- **Standard Deviation**: $\sqrt{p(1-p)}$

3. **Geometric Distribution**

- **Notation**: $X \sim \mathrm{Geo}(p)$
- **Definition**: Number of trials until first success.
- **PMF**: $P(X = x) = (1-p)^{x-1}p$, for $x = 1,2,3, \dots$
- **CDF**: $F(x) = 1-(1-p)^x$
- **Expectation**: $E[X] = \frac{1}{p}$
- **Variance**: $\mathrm{Var}(X) = \frac{1-p}{p^2}$
- **Standard Deviation**: $\sqrt{\frac{1-p}{p^2}}$

4. **Binomial Distribution**

- **Notation**: $X \sim \mathrm{Bin}(n, p)$
- **Definition**: Number of successes in $n$ independent Bernoulli trials.
- **PMF**: $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$, for $k = 0,1, \dots, n$
- **CDF**: Sum of PMFs up to $k$
- **Expectation**: $E[X] = np$
- **Variance**: $\mathrm{Var}(X) = np(1-p)$
- **Standard Deviation**: $\sqrt{np(1-p)}$

5. **Poisson Distribution**

- **Notation**: $X \sim$ Poisson( $\lambda$ )
- **Definition**: Models number of events in a fixed interval with average rate $\lambda$.
- **PMF**: $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$, for $k = 0,1,2, \dots$
- **CDF**: Sum of PMFs up to $k$
- **Expectation**: $E[X] = \lambda$
- **Variance**: $\mathrm{Var}(X) = \lambda$
- **Standard Deviation** $\sqrt{\lambda}$

### 6.2 Continuous Distributions

1. **Continuous Uniform Distribution**

- **Notation**: $X \sim U(a, b)$
- **Definition**: Uniform density over interval $[a, b]$
- **PDF**: $f(x) = \frac{1}{b-a}$, for $a \leq x \leq b$
- **CDF**: $F(x) = \frac{x-a}{b-a}$, for $a \leq x \leq b$
- **Expectation**: $E[X] = \frac{a+b}{2}$
- **Variance**: $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$
- **Standard Deviation**: $\frac{b-a}{\sqrt{12}}$

2. **Exponential Distribution**

- **Notation**: $X \sim \operatorname{Exp}(\lambda)$
- **Definition**: Time between events in a Poisson process.
- **PDF**: $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$
- **CDF**: $F(x) = 1 - e^{-\lambda x}$
- **Expectation**: $E[X] = \frac{1}{\lambda}$
- **Variance**: $\operatorname{Var}(X) = \frac{1}{\lambda^2}$
- **Standard Deviation**: $\frac{1}{\lambda}$

3. **Normal Distribution**

- **Notation**: $X \sim N(\mu, \sigma^2)$
- **Definition**: Bell-shaped symmetric distribution.
- **PDF**:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- **CDF**: No closed form; denoted by $\Phi(x)$
- **Expectation**: $E[X] = \mu$
- **Variance**: $\operatorname{Var}(X) = \sigma^2$
- **Standard Deviation**: $\sigma$

4. **Standard Normal Distribution**

- **Notation**: $Z \sim N(0,1)$
- **Definition**: Normal distribution with mean 0 and variance 1.
- **PDF**:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

- **CDF**: Denoted by $\Phi(z)$
- **Expectation**: $E[Z] = 0$
- **Variance**: $\operatorname{Var}(Z) = 1$
- **Standard Deviation**: 1

### 6.3 Moment Generating Function (MGF)

- The Moment Generating Function of a random variable $\mathbf{X}$ is defined as:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Where $t$ is a real number such that the expectation exists.

**Why MGF?**

- Helps in finding moments (mean, variance, etc.) of a distribution.
- Uniquely determines the distribution (if it exists in an open interval around 0 ).
- Simplifies calculations for the sum of independent random variables.

**How to find moments using MGF:**

- The $\mathbf{n}$-th moment about the origin is obtained by differentiating the MGF $\mathbf{n}$ times:

$$\mu_n' = \left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0}$$

- Example:
- Mean: $\mu = M_X'(0)$
- Variance: $\sigma^2 = M_X''(0) - (M_X'(0))^2$

**Discrete Distributions:**

1. **Uniform (Discrete)** $U(a,b)$

$$M_X(t) = \frac{1}{b-a+1} \sum_{x=a}^{b} e^{tx}$$

2. **Bernoulli(p)**

$$M_X(t) = (1-p) + pe^t$$

3. **Geometric(p)** (Number of trials until first success, starting from 1)

$$M_X(t) = \frac{pe^t}{1-(1-p)e^t}, \text{ for } t < -\ln(1-p)$$

4. **Binomial( $\mathbf{n}, \mathbf{p}$ )**

$$M_X(t) = [(1-p) + pe^t]^n$$

**Negative Binomial(r, p)** (Number of trials until r successes)

$$M_X(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^r, t < -\ln(1-p)$$

5. **Poisson** $(\lambda)$

$$M_X(t) = \exp(\lambda(e^t - 1))$$

**Continuous Distributions:**

1. **Uniform (Continuous)** $U(a, b)$

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, t \neq 0$$

2. **Exponential** $(\lambda)$

$$M_X(t) = \frac{\lambda}{\lambda - t}, t < \lambda$$

3. $\mathbf{Normal}(\mu, \sigma^2)$

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$$

4. **Standard Normal** $\mathcal{N}(0, 1)$

$$M_X(t) = \exp\left(\frac{1}{2}t^2\right)$$

## CALCULUS GATE DA

### 1. Functions of a single variable

A function f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the domain of the function. The set of outputs is called the range of the function.
OR
A function is a rule that maps a number to another unique number.

### 1.1 Domain

The domain of a function is the set of possible inputs.

### 1.2 Range

The range of a function is the set of corresponding outputs.

### 1.3 Image and Pre-Image

b is the image of a and a is the pre-image of b if f(a) = b

### 1.4 Even and Odd function

Even Function:

- **Definition:** A function f(x) is even if f(-x) = f(x) for all x in its domain.
- **Symmetry:** The graph of an even function is symmetric about the y-axis.
- **Examples:** $f(x) = x^2$, f(x) = cos(x), f(x) = |x|.

Odd Function:

- **Definition:** A function f(x) is odd if f(-x) = -f(x) for all x in its domain.
- **Symmetry:** The graph of an odd function is symmetric about the origin.
- **Examples:** $f(x) = x^3$, f(x) = sin(x), f(x) = x.

### 1.5 Function Arithmetic

Function arithmetic refers to performing basic arithmetic operations (addition, subtraction, multiplication, and division) on functions. If f(x) and g(x) are two functions, we can define new functions by combining them using these operations.

- The sum of f and g, denoted f + g, is the function defined by the formula

  $$(f + g)(x) = f(x) + g(x)$$

- The difference of f and g, denoted f − g, is the function defined by the formula

  $$(f - g)(x) = f(x) - g(x)$$

- The product of f and g, denoted fg, is the function defined by the formula

  $$(fg)(x) = f(x)g(x)$$

- The quotient of f and g, denoted $\frac{f}{g}$, is the function defined by the formula

  $$\frac{f}{g}(x) = \frac{f(x)}{g(x)},$$

  provided g(x) ≠ 0.

### 1.6 Composition functions

Let f : A → B and g : B → C be two functions.
Then the composition of f and g, denoted by g ∘ f, is defined as the function
g ∘ f : A → C
given by g ∘ f (x) = g(f (x)), ∀ x ∈ A.
The order of function is an important thing while dealing with the composition of functions since (f ∘ g) (x) is not equal to (g ∘ f) (x).

### 1.7 Types of Functions

Functions can be classified based on how elements of the domain are mapped to the codomain.

#### 1. One-One Function (Injective)

A function f:A→B is called **one-one** if different elements in the domain map to different elements in the codomain.

Mathematically,
$f(x1)=f(x2) \Rightarrow x1=x2$

#### 2. Onto Function (Surjective)

A function f:A→B is called **onto** if **every element of the codomain** is the image of **at least one element** in the domain.

Formally,
$\forall y \in B, \exists x \in A$ such that f(x)=y

#### 3. One-One Onto Function (Bijective)

A function is called **bijective** if it is both **one-one and onto**.

Every element of the domain maps to a unique element in the codomain, and every element of the codomain is covered.

### 1.8 Inverse function

Let $f: A \rightarrow B$ be a function from set $A$ (domain) to set $B$ (codomain).
The inverse of a function reverses this mapping - it maps each element of the codomain back to its original input from the domain.
If $f(x) = y$, then the inverse function $f^{-1}$ satisfies:

$$f^{-1}(y) = x$$

That is:

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(x)) = x$$

## 2. Limit

The limit of a function describes the behavior of the function as the input approaches a particular value.
If $f(x)$ becomes arbitrarily close to a number $L$ as $x$ approaches $a$, we write:

$$\lim_{x \to a} f(x) = L$$

This means the function values approach $L$, even if $f(a)$ is undefined or not equal to $L$.

### 2.1 Left-Hand and Right-Hand Limits

- Left-hand limit: $\lim_{x \to a^-} f(x)$
- Right-hand limit: $\lim_{x \to a^+} f(x)$

The limit exists at $x = a$ only if both left-hand and right-hand limits exist and are equal.

**2.2** Limit Laws Suppose that $c$ is a constant and the limits
$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$

exist. Then

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[cf(x)] = c\lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

### 2.3 L'Hôpital's Rule

**If**

- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ (or both limits are $\infty$ ),
- and the derivatives $f'(x)$ and $g'(x)$ exist near $x = a$,

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

## 3. Continuity of a Function

### 3.1 Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

That is, the limit of the function as $x$ approaches $a$ exists and is equal to the actual value of the function at that point.
Conditions for Continuity at $\mathbf{x} = \mathbf{a}$
A function $f(x)$ is continuous at $x = a$ if and only if all of the following hold:

1. $f(a)$ is defined
   The function has a value at $x = a$.
2. $\lim_{x \to a} f(x)$ exists
   The left-hand and right-hand limits both exist and are equal.
3. $\lim_{x \to a} f(x) = f(a)$
   The function's limit at $a$ equals its value at $a$.

### Conditions for Continuity at x=a

### 3.2 Continuity in an Interval

3.2 Continuity in an Interval
A function $f(x)$ is said to be continuous on an interval if it is continuous at every point within that interval.

- Open interval $(a, b)$ : Continuous at all points in the interval.
- Closed interval $[a, b]$ :
- Continuous at every point in $(a, b)$,
- Right-continuous at $a$ : $\lim_{x \to a^+} f(x) = f(a)$
- Left-continuous at $b$: $\lim_{x \to b^-} f(x) = f(b)$

### 3.3 Types of Discontinuities

If any of the three conditions for continuity fail, the function is discontinuous at that point. Common types:

1. Removable Discontinuity:

Limit exists, but $f(a)$ is either undefined or not equal to the limit.

2. Jump Discontinuity:

Left-hand and right-hand limits exist but are not equal.

3. Infinite Discontinuity:

Limit does not exist because function tends to $\infty$ or $-\infty$.

## 4. Differentiability of a Function

A function $f(x)$ is said to be differentiable at a point $x = a$ if the derivative exists at that point. This means the function has a well-defined, unique tangent at that point, and its rate of change is smooth (no sharp turns or corners).

The derivative of $f(x)$ at $x = a$ is defined as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists and is finite, then $f$ is differentiable at $x = a$.

### 4.1 Left and Right Derivative

To verify differentiability at $x = a$, both the left-hand derivative (LHD) and right-hand derivative ( R must exist and be equal:

- Right-hand derivative:
$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}$$
- Left-hand derivative:
$$\lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}$$

If:

$$\text{LHD} = \text{RHD}$$

Then $f$ is differentiable at $x = a$

### 4.2 Differentiability Implies Continuity

- If a function is **differentiable at a point**, then it is also **continuous at that point**.
- But the **converse is not true**: a function can be **continuous but not differentiable** (e.g., at a sharp corner or cusp).

### 4.3 Summary:

| Property | Condition |
|---|---|
| Limit exists | $\lim_{x\to a} f(x) = L$ |
| Continuity | $\lim_{x\to a} f(x) = f(a)$ |
| Differentiability | $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists and finite |

## 5. Taylor Series

The **Taylor series** of a function is an infinite series that represents a function as a sum of its derivatives at a single point.

If a function $f(x)$ is infinitely differentiable at a point $a$, then the Taylor series of $f(x)$ about $x = a$ is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

### 5.1 Special Case: Maclaurin Series

If $a = 0$, the Taylor series becomes the Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

### Use of Taylor Series:

- Approximate complex functions using polynomials.
- Estimate values of functions near a point.
- Analyze error using **Taylor remainder term**.

### 5.2 Error in Taylor Polynomial (Remainder Term)

When we approximate a function f(x)f(x)f(x) using a Taylor polynomial of degree nnn, the approximation is not exact unless the function is a polynomial of degree ≤n\leq n≤n.

The **error** or **remainder** tells us **how far off** the approximation is from the true value of the function.

Let $f$ be $(n+1)$ times differentiable on an interval around $x = a$. Then for any $x$ in that interval,

$$f(x) = P_n(x) + R_n(x)$$

Where:

- $P_n(x)$ is the Taylor polynomial of degree $n$ centered at $x = a$

- $R_n(x)$ is the remainder (error) term

### 5.3 Lagrange's Form of the Remainder:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

- $c$ is some number between $a$ and $x$
- This gives the exact form of the error

## 6. Local and Global Maxima/Minima

These are points where a function reaches its **highest or lowest values**, either in a **small neighbourhood** (local) or over the **entire domain** (global).

### Local Maximum:

A function $f(x)$ has a local maximum at $x = a$ if:

$f(a) \geq f(x)$ for all $x$ near $a$

- The function reaches a peak locally, i.e., within a small interval around $a$.
- Derivative test: if $f'(a) = 0$ and $f''(a) < 0$, it's a local max.

### Local Minimum:

A function $f(x)$ has a local minimum at $x = a$ if:

$f(a) \leq f(x)$ for all $x$ near $a$

- The function reaches a dip locally.
- Derivative test: if $f'(a) = 0$ and $f''(a) > 0$, it's a local min.

### Global Maximum:

A function $f(x)$ has a global (absolute) maximum at $x = a$ if:

$f(a) \geq f(x)$ for all $x$ in the domain

- It's the highest value the function ever reaches.

### Global Minimum:

A function $f(x)$ has a global (absolute) minimum at $x = a$ if:

$f(a) \leq f(x)$ for all $x$ in the domain

- It's the lowest value the function reaches overall.

### 6.1 Finding Maxima and MinimCritical Points:

A point $x = c$ is called a critical point of $f(x)$ if either:

- $f'(c) = 0$ (stationary point), or
- $f'(c)$ does not exist.

### First Derivative Test:

- If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum.
- If $f'(x)$ changes sign from negative to positive, then $f(c)$ is a local minimum.

### Second Derivative Test:

- If $f'(c) = 0$, then:
- If $f''(c) > 0$ : local minimum at $c$
- If $f''(c) < 0$ : local maximum at $c$
- If $f''(c) = 0$ : test is inconclusive

### Critical Points:

A point $x = c$ is called a critical point of $f(x)$ if either:

- $f'(c) = 0$ (stationary point), or
- $f'(c)$ does not exist.

### First Derivative Test:

- If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum.
- If $f'(x)$ changes sign from negative to positive, then $f(c)$ is a local minimum.

### Second Derivative Test:

- If $f'(c) = 0$, then:
- If $f''(c) > 0$ : local minimum at $c$
- If $f''(c) < 0$ : local maximum at $c$
- If $f''(c) = 0$ : test is inconclusive

## 7. Indefinite Integral

- **Definition:**

The integral of a function $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x)$$

Denoted as:

$$\int f(x)dx = F(x) + C$$

where $C$ is the constant of integration.

- **Basic Properties:**

1. $\int 0dx = C$
2. $\int kf(x)dx = k\int f(x)dx$ (constant multiple rule)
3. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
4. $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

- **Standard Results:**

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C\ (n \neq -1)$
2. $\int \frac{1}{x}dx = \ln|x| + C$
3. $\int e^x dx = e^x + C$
4. $\int a^x dx = \frac{a^x}{\ln a} + C, (a > 0, a \neq 1)$
5. $\int \sin x dx = -\cos x + C$
6. $\int \cos x dx = \sin x + C$
7. $\int \sec^2 x dx = \tan x + C$
8. $\int \csc^2 x dx = -\cot x + C$
9. $\int \sec x \tan x dx = \sec x + C$
10. $\int \csc x \cot x dx = -\csc x + C$

## 9. Definite Integral

- **Definition:**

A definite integral of $f(x)$ from $a$ to $b$ :

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$.

- Interpretation:

Represents the signed area under the curve $y = f(x)$ between $x = a$ and $x = b$.

- **Properties:**

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
5. If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq 0$.
6. If $f(x)$ is continuous and $f(x) = f(a + b - x)$, then
$$\int_a^b f(x)dx = 2\int_a^{\frac{a+b}{2}} f(x)dx$$

- **Fundamental Theorem of Calculus (FTC):**

1. If $F(x)$ is antiderivative of $f(x)$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$
2. $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$

# STAR MENTOR CS/DA

**KHALEEL SIR**
ALGORITHM & OS
29 YEARS OF TEACHING EXPERIENCE

**SATISH SIR**
DISCRETE MATHEMATICS
BE in IT from MUMBAI UNIVERSITY

**VIJAY SIR**
DBMS & COA
M. TECH FROM NIT
14+ YEARS EXPERIENCE

**SAKSHI MA'AM**
ENGINEERING MATHEMATICS
IIT ROORKEE ALUMNUS

**AVINASH SIR**
APTITUDE
10+ YEARS OF TEACHING EXPERIENCE

**CHANDAN SIR**
DIGITAL LOGIC
GATE AIR 23 & 26 / EX-ISRO

**MALLESHAM SIR**
M.TECH FROM IIT BOMBAY
AIR – 114, 119, 210 in GATE
(CRACKED GATE 8 TIMES)
14+ YEARS EXPERIENCE

**PARTH SIR**
DA
IIIT BANGALORE ALUMNUS
FORMER ASSISTANT PROFESSOR

**SHAILENDER SIR**
C PROGRAMMING & DATA STRUCTURE
M.TECH in Computer Science
15+ YEARS EXPERIENCE

**AJAY SIR**
PH.D. IN COMPUTER SCIENCE
12+ YEARS EXPERIENCE