GeeksforGeeks

GATE फर्रें

DATA SCIENCE & ARTIFIACL INTELLIGENCE (DA)

CALCULUS

SHORT NOTES

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1. Functions of a single variable

A function f consists of a set of inputs, a set of outputs, and a rule for assigning each input to exactly one output. The set of inputs is called the domain of the function. The set of outputs is called the range of the function.
OR
A function is a rule that maps a number to another unique number.

1.1 Domain

The domain of a function is the set of possible inputs.

1.2 Range

The range of a function is the set of corresponding outputs.

1.3 Image and Pre-Image

b is the image of a and a is the pre-image of b if f(a) = b

1.4 Even and Odd function

Even Function:

- **Definition:** A function f(x) is even if f(-x) = f(x) for all x in its domain.
- **Symmetry:** The graph of an even function is symmetric about the y-axis.
- **Examples:** $f(x) = x^2$, f(x) = cos(x), f(x) = |x|.

Odd Function:

- **Definition:** A function f(x) is odd if f(-x) = -f(x) for all x in its domain.
- **Symmetry:** The graph of an odd function is symmetric about the origin.
- **Examples:** $f(x) = x^3$, f(x) = sin(x), f(x) = x.

1.5 Function Arithmetic

Function arithmetic refers to performing basic arithmetic operations (addition, subtraction, multiplication, and division) on functions. If f(x) and g(x) are two functions, we can define new functions by combining them using these operations.

- The sum of f and g, denoted f + g, is the function defined by the formula
  $$(f + g)(x) = f(x) + g(x)$$
- The difference of f and g, denoted f − g, is the function defined by the formula
  $$(f - g)(x) = f(x) - g(x)$$
- The product of f and g, denoted fg, is the function defined by the formula
  $$(fg)(x) = f(x)g(x)$$
- The quotient of f and g, denoted $\frac{f}{g}$, is the function defined by the formula
  $$\frac{f}{g}(x) = \frac{f(x)}{g(x)},$$

provided g(x) ≠ 0.

1.6 Composition functions

Let f : A → B and g : B → C be two functions.
Then the composition of f and g, denoted by g ∘ f, is defined as the function
$g \circ f : A \to C$
given by $g \circ f (x) = g(f (x)), \forall x \in A$.
The order of function is an important thing while dealing with the composition of functions since (f ∘ g) (x) is not equal to (g ∘ f) (x).

1.7 Types of Functions

Functions can be classified based on how elements of the domain are mapped to the codomain.

1. One-One Function (Injective)

A function f:A→B is called **one-one** if different elements in the domain map to different elements in the codomain.

Mathematically,
$f(x1)=f(x2) \Rightarrow x1=x2$

2. Onto Function (Surjective)

A function f:A→B is called **onto** if **every element of the codomain** is the image of **at least one element** in the domain.

Formally,
$\forall y \in B, \exists x \in A$ such that $f(x)=y$

3. One-One Onto Function (Bijective)

A function is called **bijective** if it is both **one-one and onto**.
Every element of the domain maps to a unique

element in the codomain, and every element of the codomain is covered.

1.8 Inverse function
Let $f: A \to B$ be a function from set A (domain) to set B (codomain).
The inverse of a function reverses this mapping - it maps each element of the codomain back to its original input from the domain.
If $f(x) = y$, then the inverse function f^{-1} satisfies:

$$f^{-1}(y) = x$$

That is:

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(x)) = x$$

2. Limit
The limit of a function describes the behavior of the function as the input approaches a particular value.
If $f(x)$ becomes arbitrarily close to a number L as x approaches a, we write:

$$\lim_{x \to a} f(x) = L$$

This means the function values approach L, even if $f(a)$ is undefined or not equal to L.

2.1 Left-Hand and Right-Hand Limits

- Left-hand limit: $\lim_{x \to a^-} f(x)$
- Right-hand limit: $\lim_{x \to a^+} f(x)$

The limit exists at $x = a$ only if both left-hand and right-hand limits exist and are equal.

2.2 Limit Laws Suppose that c is a constant and the limits
$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c\lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

2.3 L'Hôpital's Rule

If

- $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ (or both limits are ∞),
- and the derivatives $f'(x)$ and $g'(x)$ exist near $x = a$,

Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

3. Continuity of a Function

3.1 Continuity of a Function at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ if:

$$\lim_{x \to a} f(x) = f(a)$$

That is, the limit of the function as x approaches a exists and is equal to the actual value of the function at that point.
Conditions for Continuity at $\mathbf{x = a}$
A function $f(x)$ is continuous at $x = a$ if and only if all of the following hold:

1. $f(a)$ is defined
   The function has a value at $x = a$.
2. $\lim_{x \to a} f(x)$ exists
   The left-hand and right-hand limits both exist and are equal.
3. $\lim_{x \to a} f(x) = f(a)$
   The function's limit at a equals its value at a.

Conditions for Continuity at x=a

3.2 Continuity in an Interval
3.2 Continuity in an Interval
A function $f(x)$ is said to be continuous on an interval if it is continuous at every point within that interval.

- Open interval (a, b) : Continuous at all points in the interval.
- Closed interval $[a, b]$:
- Continuous at every point in (a, b),
- Right-continuous at a : $\lim_{x \to a^+} f(x) = f(a)$

- Left-continuous at b: $\lim_{x \to b^-} f(x) = f(b)$

3.3 Types of Discontinuities
If any of the three conditions for continuity fail, the function is discontinuous at that point. Common types:

1. Removable Discontinuity:

Limit exists, but $f(a)$ is either undefined or not equal to the limit.
2. Jump Discontinuity:
Left-hand and right-hand limits exist but are not equal.
3. Infinite Discontinuity:
Limit does not exist because function tends to ∞ or $-\infty$.

4. Differentiability of a Function
A function $f(x)$ is said to be differentiable at a point $x = a$ if the derivative exists at that point.
This means the function has a well-defined, unique tangent at that point, and its rate of change is smooth (no sharp turns or corners).
The derivative of $f(x)$ at $x = a$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If this limit exists and is finite, then f is differentiable at $x = a$.

4.1 Left and Right Derivative
To verify differentiability at $x = a$, both the left-hand derivative (LHD) and right-hand derivative (R must exist and be equal:

- Right-hand derivative:

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

- Left-hand derivative:

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

If:

$$\text{LHD} = \text{RHD}$$

Then f is differentiable at $x = a$

4.2 Differentiability Implies Continuity
- If a function is **differentiable at a point**, then it is also **continuous at that point**.
- But the **converse is not true**: a function can be **continuous but not differentiable** (e.g., at a sharp corner or cusp).

4.3 Summary:

Property	Condition
Limit exists	$\lim_{x \to a} f(x) = L$
Continuity	$\lim_{x \to a} f(x) = f(a)$
Differentiability	$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists and finite

5. Taylor Series
The **Taylor series** of a function is an infinite series that represents a function as a sum of its derivatives at a single point.

If a function $f(x)$ is infinitely differentiable at a point a, then the Taylor series of $f(x)$ about $x = a$ is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

5.1 Special Case: Maclaurin Series

If $a = 0$, the Taylor series becomes the Maclaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Use of Taylor Series:
- Approximate complex functions using polynomials.
- Estimate values of functions near a point.
- Analyze error using **Taylor remainder term**.

5.2 Error in Taylor Polynomial (Remainder Term)
When we approximate a function f(x)f(x)f(x) using a Taylor polynomial of degree nnn, the approximation is not exact unless the function is a polynomial of degree ≤n\leq n≤n.
The **error** or **remainder** tells us **how far off** the approximation is from the true value of the function.
Let f be $(n+1)$ times differentiable on an interval around $x = a$. Then for any x in that interval,

$$f(x) = P_n(x) + R_n(x)$$

Where:
- $P_n(x)$ is the Taylor polynomial of degree n centered at $x = a$
- $R_n(x)$ is the remainder (error) term

5.3 Lagrange's Form of the Remainder:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

- c is some number between a and x
- This gives the exact form of the error

6. Local and Global Maxima/Minima

These are points where a function reaches its **highest or lowest values**, either in a **small neighbourhood** (local) or over the **entire domain** (global).

Local Maximum:

A function $f(x)$ has a local maximum at $x = a$ if:

$$f(a) \geq f(x) \text{ for all } x \text{ near } a$$

- The function reaches a peak locally, i.e., within a small interval around a.
- Derivative test: if $f'(a) = 0$ and $f''(a) < 0$, it's a local max.

Local Minimum:

A function $f(x)$ has a local minimum at $x = a$ if:

$$f(a) \leq f(x) \text{ for all } x \text{ near } a$$

- The function reaches a dip locally.
- Derivative test: if $f'(a) = 0$ and $f''(a) > 0$, it's a local min.

Global Maximum:

A function $f(x)$ has a global (absolute) maximum at $x = a$ if:

$$f(a) \geq f(x) \text{ for all } x \text{ in the domain}$$

- It's the highest value the function ever reaches.

Global Minimum:

A function $f(x)$ has a global (absolute) minimum at $x = a$ if:

$$f(a) \leq f(x) \text{ for all } x \text{ in the domain}$$

- It's the lowest value the function reaches overall.

6.1 Finding Maxima and MinimCritical Points:

A point $x = c$ is called a critical point of $f(x)$ if either:

- $f'(c) = 0$ (stationary point), or
- $f'(c)$ does not exist.

First Derivative Test:

- If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum.
- If $f'(x)$ changes sign from negative to positive, then $f(c)$ is a local minimum.

Second Derivative Test:

- If $f'(c) = 0$, then:
- If $f''(c) > 0$: local minimum at c
- If $f''(c) < 0$: local maximum at c
- If $f''(c) = 0$: test is inconclusive

Critical Points:

A point $x = c$ is called a critical point of $f(x)$ if either:

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Steps for Solving Optimization Problems:

1. **Understand the problem**: Identify the quantity to be maximized or minimized.
2. **Formulate a function** f(x) that represents that quantity.
3. **Find the domain** of f(x) (often based on physical or geometric constraints).
4. **Find critical points** by solving 0f'(x)=0.
5. **Use first or second derivative test** to determine maxima/minima.
6. **Evaluate function at endpoints** (if applicable) and at critical points.
7. **Choose the point** that gives the required maximum or minimum.

7. Optimization Involving a Single Variable

Optimization is the process of **finding the maximum or minimum value** of a function under certain conditions.